

AN  $L^2$ -INDEX THEOREM FOR DIRAC OPERATORS ON  $S^1 \times \mathbb{R}^3$ 

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ABSTRACT. An expression is found for the  $L^2$ -index of a Dirac operator coupled to a connection on a  $U_n$  vector bundle over  $S^1 \times \mathbb{R}^3$ . Boundary conditions for the connection are given which ensure the coupled Dirac operator is Fredholm. Callias' index theorem is used to calculate the index when the connection is independent of the coordinate on  $S^1$ . An excision theorem due to Gromov, Lawson, and Anghel reduces the index theorem to this special case. The index formula can be expressed using the adiabatic limit of the  $\eta$ -invariant of a Dirac operator canonically associated to the boundary. An application of the theorem is to count the zero modes of the Dirac operator in the background of a caloron (periodic instanton).

## 1. INTRODUCTION

Let  $X$  be a compact, oriented smooth manifold with boundary  $\partial X$ , and let  $X^\circ = X \setminus \partial X$  be the corresponding open manifold. Let  $g$  be a complete Riemannian metric on  $X^\circ$  and let  $E \rightarrow X^\circ$  be a complex vector bundle with a hermitian structure and unitary connection  $A$ . If  $X$  is a spin-manifold, we can introduce the coupled Dirac operator

$$D_A : C^\infty(X^\circ, S \otimes E) \rightarrow C^\infty(X^\circ, S \otimes E)$$

and it is natural to attempt to obtain first, conditions on  $A$  and  $g$  near  $\partial X$  which ensure that  $D_A$  is a Fredholm operator on  $L^2$ , and second, to obtain a formula for the  $L^2$ -index. Since with standard conventions  $D_A$  is self-adjoint, we must explain how one obtains interesting  $L^2$ -index problems from it.

If  $\dim X$  is *even*, then the total spin-bundle decomposes as  $S = S^+ \oplus S^-$  and so  $D_A$  gives rise to the 'chiral' Dirac operators

$$D_A^\pm : C^\infty(X^\circ, S^\pm \otimes E) \rightarrow C^\infty(X^\circ, S^\mp \otimes E)$$

which have equal and opposite  $L^2$ -indices. If the geometry near  $\partial X$  is restricted so that  $g$  is a *b-metric*, then the celebrated index theorem of Atiyah, Patodi and Singer [3] gives a formula for the  $L^2$ -index of  $D_A^+$  when this operator is Fredholm. (The notion of *b-metric* was not used explicitly in [3]; they worked with the equivalent idea of  $X^\circ$  having cylindrical ends, a simple condition on  $g$  near  $\partial X$ . The APS index theorem is discussed from the point of view of *b-metrics* in [15].) The APS theorem expresses the  $L^2$ -index of  $D_A^+$  as a sum of two terms, one an integral of characteristic classes over  $X$  and a boundary contribution involving the famous  $\eta$ -invariant of  $\partial X$ .

By contrast, when  $\dim X$  is *odd*, an interesting index-problem arises for operators of the form

$$D_{A,\Phi} := D_A + 1 \otimes \Phi : C^\infty(X^\circ, S \otimes E) \rightarrow C^\infty(X^\circ, S \otimes E) \quad (1)$$

where  $\Phi$  is a suitable skew-adjoint endomorphism of  $E$ . According to work of Callias, Anghel and Råde [7] [2] [17],  $D_{A,\Phi}$  is Fredholm in  $L^2$  under mild conditions on  $\Phi$ , the most important being that it be invertible on  $\partial X$ . Then, with no further

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restrictions on the geometry of  $g$  near  $\partial X$ , the index is given by integrating over  $\partial X$  a certain characteristic class constructed from  $\Phi|_{\partial X}$  and  $E \otimes S|_{\partial X}$ . We shall refer to this as the CAR index theorem in this paper, even though the result is closely related to pre-existing index theorems (cf. [17] for a discussion of this point).

The purpose of this note is to state and prove an index formula for  $D_A^+$  when  $X$  is even-dimensional, but the geometry near the boundary is not that of a  $b$ -metric. We shall restrict ourselves to a very simple special case: we take  $(X^\circ, g)$  to be isometric to  $S^1 \times \mathbb{R}^3$ , with a standard flat product metric. Then  $X$  is diffeomorphic to  $S^1 \times \overline{B}^3$  where  $\overline{B}^3$  is the closed unit ball in  $\mathbb{R}^3$ . Despite its simplicity, this example already leads to an interesting index theorem, thereby answering a question posed by Mazzeo and Melrose in their study of  $\Phi$ -pseudodifferential operators [14], at least in this very special case. (This  $\Phi$  stands for ‘fibred-cusp’ and has nothing to do with the  $\Phi$  in (1). This notational clash is unfortunate but seems unlikely to lead to serious confusion.) We should note also that an  $L^2$ -index theorem for the coupled Dirac operator over  $S^1 \times S^1 \times \mathbb{R}^2$  has been obtained by Jardim [12]. This is another natural example of a  $\Phi$ -geometry, but with fibre dimension 2 rather than 1.

The index theorem on  $S^1 \times \mathbb{R}^3$  is important in the study of self-dual calorons (periodic instantons) of which more will be said in §1.3.

**1.1. Statement of results.** The following notation will be used throughout this paper: let  $X = S^1 \times \overline{B}^3$ ,  $S_\infty^2 = \partial \overline{B}^3$ , so that  $\partial X = S^1 \times S_\infty^2$ ; let  $p : X \rightarrow \overline{B}^3$  be the projection. Let the metric  $g$  on  $X^\circ$  be the standard flat product metric on  $S^1 \times \mathbb{R}^3$  giving the circle a length  $2\pi/\mu_0$ , where  $\mu_0 > 0$ . Let  $z \in \mathbb{R}/(2\pi/\mu_0)\mathbb{Z}$  be a standard coordinate on the circle, and  $x_1, x_2, x_3$  standard coordinates on  $\mathbb{R}^3$ . Finally, fix an orientation on  $X$  by decreeing that the ordered basis  $(dz, dx_1, dx_2, dx_3)$  be positive.

Let  $\mathbb{E} \rightarrow X$  be a smooth  $U_n$ -bundle and let  $\mathbb{A}$  be a smooth unitary connection in  $\mathbb{E}$ .  $\mathbb{A}$  will be identified with the corresponding covariant derivative operator  $\nabla$ , which has components  $\nabla_z, \nabla_1, \nabla_2, \nabla_3$  in the frame  $(\partial_z, \partial_1, \partial_2, \partial_3)$ . Each of the two spin-bundles  $S^\pm$  over  $X^\circ$  can be identified with  $p^*S_{(3)}$ , where  $S_{(3)}$  is the spin-bundle of  $\mathbb{R}^3$ . This is a complex vector bundle of rank 2 and comes equipped with skew-adjoint Clifford multiplication operators  $e_1, e_2, e_3$  associated with  $\partial_1, \partial_2, \partial_3$ . The two coupled Dirac operators over  $X^\circ$  can now be written

$$D_{\mathbb{A}}^\pm = \pm \nabla_z + D_A : C^\infty(X^\circ, p^*S_{(3)} \otimes \mathbb{E}) \rightarrow C^\infty(X^\circ, p^*S_{(3)} \otimes \mathbb{E}) \quad (2)$$

where  $D_A = \sum_1^3 e_j \nabla_j$ . The first term in (2) operates on sections of the tensor product by  $\nabla_z(p^*(s) \otimes u) = p^*(s) \otimes \nabla_z u$  for any  $s \in C^\infty(\mathbb{R}^3, S_{(3)})$  and  $u \in C^\infty(X^\circ, \mathbb{E})$ .

Note that the ‘3+1’ decomposition of  $D_{\mathbb{A}}^\pm$  in (2) depends only upon the product structure of  $X$ ; we introduced bases only for ease of presentation. We now focus on  $D_{\mathbb{A}}^+$ ; by abuse of notation, denote by the same symbol the extension of  $D_{\mathbb{A}}^+$  to Sobolev spaces

$$D_{\mathbb{A}}^+ : W^1(X, S^+ \otimes \mathbb{E}) \rightarrow W^0(X, S^- \otimes \mathbb{E}), \quad (3)$$

where  $W^k$  is the space of sections with  $k$  derivatives in  $L^2$ , the latter space being defined in terms of the metric  $g$ .

In order to fix boundary conditions that make (3) a Fredholm operator it is convenient to fix a trivialization of  $\mathbb{E}|_{\partial X}$ :

**Definition 1.** A framing of  $\mathbb{E}$  is a choice of trivialization  $f$  of  $\mathbb{E}|_{\partial X}$ . The pair  $(\mathbb{E}, f)$  is called a framed bundle.

As we shall see in §2.1, framed bundles of rank  $\geq 2$  are classified by an integral topological invariant analogous to the second Chern class, which we shall denote by  $c_2(\mathbb{E}, f)[X]$ .

For later convenience, we identify the trivial bundle that is implicit in Definition 1 with  $p^*E_\infty$ , where  $E_\infty$  denotes the trivial bundle over  $S_\infty^2$ . Now we can write down boundary conditions for  $\mathbb{A}$ :

**Definition 2.** Let  $\mathbb{A}$  be a connection on a framed bundle  $(\mathbb{E}, f)$ , smooth up to the boundary of  $X$ , let  $A_\infty$  be a  $U_n$ -connection on  $E_\infty$  and let  $\Phi_\infty$  be a skew-adjoint endomorphism of  $E_\infty$ .

(i)  $\mathbb{A}$  is called a *caloron configuration* framed by  $(A_\infty, \Phi_\infty)$  if

$$\mathbb{A} = p^*A_\infty + p^*\Phi_\infty dz$$

on  $\partial X$ , where  $f$  has been used to identify  $\mathbb{E}|_{\partial X}$  with  $p^*E_\infty$ .

(ii) The pair  $(A_\infty, \Phi_\infty)$  is called *admissible* if  $\nabla_\infty \Phi_\infty = 0$ , where  $\nabla_\infty$  is the covariant derivative operator induced by  $A_\infty$  on  $\text{End}(E_\infty)$ .

Our first main result asserts that the Fredholm properties of (3) are entirely determined by  $\Phi_\infty$ :

**Theorem 1.** Let  $\mathbb{A}$  be a caloron configuration framed by an admissible pair  $(A_\infty, \Phi_\infty)$ . Then the operator in (3) is Fredholm if and only if  $1 - \exp(2\pi\Phi_\infty/\mu_0)$  is invertible.

If  $(A_\infty, \Phi_\infty)$  is admissible, the eigenvalues  $i\mu_1, \dots, i\mu_n$  of  $\Phi_\infty$  are constant. An equivalent formulation of this theorem is thus that (3) is *not* Fredholm if and only if there exist  $j \neq 0$  and an integer  $N$  such that  $\mu_j = N\mu_0$ .

We now come to a statement of our  $L^2$ -index theorem:

**Theorem 2.** Suppose that (3) is Fredholm. Then

$$\text{ind}(D_{\mathbb{A}}^+) = -c_2(\mathbb{E}, f)[X] - \sum_k c_1(E_{(k)}^+)[S_\infty^2] \quad (4)$$

where for each integer  $k$ ,  $E_{(k)}^+$  is the sub-bundle of  $E_\infty$  on which  $k\mu_0 - i\Phi_\infty$  is positive-definite.

It is clear that  $E_{(k)}^+$  is either 0 or  $E_\infty$  for all but a finite number of integers  $k$ . Since  $E_\infty$  is trivial, it follows that the sum on the RHS of (4) is finite.

**Corollary 1.** The index formula (4) can be re-expressed as

$$\text{ind}(D_{\mathbb{A}}^+) = \int_X \text{ch}(\mathbb{E}) - \frac{1}{2} \bar{\eta}_{\text{lim}}$$

where  $\text{ch}$  denotes the Chern character, and  $\bar{\eta}_{\text{lim}}$  is the reduced  $\eta$ -invariant of the Dirac operator on  $\partial X$  coupled to the connection  $p^*A_\infty + p^*\Phi_\infty dz$  on  $p^*E_\infty$ .

The Corollary is included as an indication of how the index theorem might be generalised and for comparison with the APS theorem—understanding  $\eta$ -invariants is not necessary for the proof of Theorem 2. We leave discussion of  $\eta$ -invariants to Appendix A, in which we sketch a proof of the Corollary.

**1.2. A sketch of the proof.** For the proof of Theorem 1, we apply general results of earlier authors. One approach is to check the conditions written down by Anghel, who has given necessary and sufficient conditions for the Dirac operator over a complete Riemannian manifold to be Fredholm in  $L^2$ . The alternative is to use the characterisation of Fredholm operators in the calculus of  $\Phi$ -pseudodifferential operators [14]. It is worth remarking that this latter approach gives necessary and sufficient conditions for any ‘natural’ operator on  $S^1 \times \mathbb{R}^3$  to be Fredholm, not only operators of Dirac type. The two proofs appear in §4.

The proof of Theorem 2 involves two main steps. The first is a calculation of the index in the case that there is a trivialization of  $\mathbb{E}$  such that  $\mathbb{A}$  is independent of  $z$ .

By Fourier analysis in the  $S^1$ -variable, the index can be identified in this case with a sum of indices of CAR-operators of the form (1). By topological arguments which we begin in §2, this calculation gives the index for any  $\mathbb{A}$  over a framed bundle with  $c_2(\mathbb{E}, f)[X] = 0$ .

The second step invokes an excision theorem for operators of Dirac type due to Anghel [1] and Gromov–Lawson [11]. In our case, this result gives  $\text{ind}(D_{\mathbb{A}}^+) - \text{ind}(D_{\mathbb{B}}^+) = -c_2(\mathbb{E}, f)[X]$  if  $\mathbb{B}$  is any caloron configuration agreeing with  $\mathbb{A}$  near  $\partial X$  but living on a new framed bundle  $(\mathbb{F}, f)$ , with  $c_2(\mathbb{F}, f)[X] = 0$ . Since we calculated  $\text{ind}(D_{\mathbb{B}}^+)$  in the first step, that completes the proof of Theorem 2. The details appear in §5.

**1.3. Application: counting the zero modes of the Dirac operator in the background of a caloron.** Connections  $\mathbb{A}$  of the type we have considered here are of interest in gauge theory, especially when they are required to satisfy the self-duality equations [9]; the term *caloron* was introduced in this context by Nahm [16]. The *Nahm transform* of a self-dual caloron  $\mathbb{A}$  is (roughly speaking) a bundle over  $\mathbb{R}$  whose fibre at  $t \in \mathbb{R}$  is the cokernel of the Dirac operator coupled to

$$\mathbb{A}_t = \mathbb{A} - itdz. \quad (5)$$

When  $\mathbb{A}$  is self-dual a Weitzenböck formula shows that  $D_{\mathbb{A}_t}^+$  is injective, so the dimension of the cokernel is given by minus the index. At the time of writing, the Nahm description has only been proved in detail for calorons on framed bundles with  $c_2(\mathbb{E}, f) = 1$  and such that the eigenbundles of  $\Phi_\infty$  are trivial [13]. Details of the full transform will be the subject of a future publication.

We want to express the index formula (4) in a form which is more familiar to mathematical physicists, and which is easier to interpret from the point of view of the Nahm transform. Fix a framed caloron configuration  $\mathbb{A}$  on a framed bundle  $(\mathbb{E}, f)$ , and define  $\mathbb{A}_t$  by (5).  $\Phi_\infty$  decomposes  $E_\infty$  into line bundles  $E_\infty = E_1 \oplus \cdots \oplus E_n$  such that  $\Phi_\infty$  acts by  $i\mu_j$  on  $E_j$ , where  $\mu_j \in \mathbb{R}$ . Define  $k_0 = c_2(\mathbb{E}, f)$  and  $k_j = c_1(E_j)[S_\infty^2]$  for  $j = 1, \dots, n$ . We can write the eigenvalues as  $\mu_j = N_j\mu_0 + \epsilon_j$  for each  $j$ , where  $0 \leq \epsilon_j < \mu_0$  and  $N_j \in \mathbb{Z}$ . Moreover, we can assume that the eigenvalues are ordered so that  $\epsilon_n \leq \epsilon_{n-1} \leq \dots \leq \epsilon_1$  (this ensures our notation matches that of Garland and Murray [9]). With this ordering, define  $m_j = k_0 + k_1 + \cdots + k_j$  for  $j = 0, \dots, n-1$ , and define intervals  $I_{j,N}$  by

$$I_{j,N} = \begin{cases} (\epsilon_1 + N\mu_0, \epsilon_n + (N+1)\mu_0) & \text{when } j = 0 \text{ and } N \in \mathbb{Z} \\ (\epsilon_{j+1} + N\mu_0, \epsilon_j + N\mu_0) & \text{for } j = 1, \dots, n-1, \text{ and } N \in \mathbb{Z}. \end{cases}$$

Then Theorem 1 implies that  $D_{\mathbb{A}_t}^+$  is Fredholm iff  $t \in \bigcup I_{j,N}$ , and Theorem 2 implies that the fibre of the Nahm transform at  $t \in I_{j,N}$  has rank  $m_j$ . Thus the rank depends periodically on  $t$  and jumps at the points  $t = N\mu_0 + \mu_j$  for  $N \in \mathbb{Z}$  and  $j = 1, \dots, n$ . This agrees with the form of the Nahm transform anticipated in [9, Section 8].

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## 2. ON THE TOPOLOGY OF CALORONS

In this section we define the invariant  $c_2(\mathbb{E}, f)[X]$  of a framed bundle over  $X$  from two points of view, the first homotopy-theoretic, the second a version of Chern–Weil theory. Proposition 1 and formula (12) will be used in the proof of Theorem 2.

**2.1. Topological classification of framed bundles.** We start with a useful way of thinking of framed bundles and calorons in terms of the ‘rectangle’  $R = [0, 2\pi/\mu_0] \times \overline{B}^3$ . There is a natural map  $R \rightarrow X$  which will be used to identify objects defined over  $X$  with corresponding objects over  $R$ . By abuse of notation we denote the second projection of  $R$  by  $p$  and we shall denote by the same symbol  $(\mathbb{E}, f)$  the pull-back to  $R$  of a framed bundle over  $X$ ; similarly for caloron configurations  $\mathbb{A}$ . In particular, when a framed bundle  $(\mathbb{E}, f)$  is transferred to  $R$ , we obtain a bundle over  $R$ , framed over  $[0, 2\pi/\mu_0] \times S_\infty^2$ , and with a ‘clutching map’

$$\phi : \mathbb{E}|_{\{0\} \times \overline{B}^3} \simeq \mathbb{E}|_{\{2\pi/\mu_0\} \times \overline{B}^3}.$$

Since  $E_\infty$  is trivial, we can regard it as the restriction to  $S_\infty^2$  of a trivial bundle  $E \rightarrow \overline{B}^3$ , say, and we can extend the framing  $f$  of  $\mathbb{E}$  to a bundle isomorphism  $F : \mathbb{E} \rightarrow p^*E$  over  $R$ . In this way  $\phi$  becomes a unitary endomorphism  $c$  of  $E$  which shall refer to as a clutching function for  $\mathbb{E}$ . Because of the periodicity,  $c$  then lies in the group

$$\mathcal{C} = \{\text{unitary automorphisms } c \text{ of } E : c|_{S_\infty^2} = 1\}.$$

Now  $\pi_0(\mathcal{C}) = \mathbb{Z}$ , for any element  $c$  extends to a continuous map from the one-point compactification  $S^3$  of  $\overline{B}^3$  into  $U_n$  and  $\pi_3(U_n) = \mathbb{Z}$ . We define  $c_2(\mathbb{E}, f)[X] = -\deg(c)$ .  $c_2(\mathbb{E}, f)[X]$  is the obstruction to extending the framing  $f$  to the interior of  $\mathbb{E}$ —an extension exists iff  $c_2(\mathbb{E}, f)[X] = 0$ .

We now reintroduce calorons, continuing to work over  $R$ , with  $\mathbb{E} = p^*E$ . Because of this identification, we can take the ‘3+1’ decomposition

$$\nabla_{\mathbb{A}} = \nabla_{A_{(z)}} + dz(\partial_z + \Phi_{(z)}) \quad (6)$$

along  $\{z\} \times \overline{B}^3$ , where  $A_{(z)}$  is a unitary connection on  $E$  and  $\Phi_{(z)}$  is a skew-adjoint endomorphism of  $E$ . Thus we have obtained from  $\mathbb{A}$  a *path*  $(A_{(z)}, \Phi_{(z)})$  in

$$\mathcal{A} = \{(A, \Phi) : A \text{ is a } U_n \text{ connection on } E,$$

$$\Phi \text{ is a skew-adjoint endomorphism of } E, (A, \Phi)|_{S_\infty^2} = (A_\infty, \Phi_\infty)\}.$$

For periodicity, the end-points of this path must be related by the clutching function  $c$ :

$$A_{(2\pi/\mu_0)} = c^*(A_{(0)}) = cA_{(0)}c^{-1} - dcc^{-1} \quad (7)$$

and

$$\Phi_{(2\pi/\mu_0)} = c^*(\Phi_{(0)}) = c\Phi_{(0)}c^{-1}. \quad (8)$$

In other words,  $\mathbb{A}$  can be identified with a loop in the quotient space  $\mathcal{A}/\mathcal{C}$ . Conversely, any such loop gives rise to a caloron configuration  $\mathbb{A}$  framed by  $(A_\infty, \Phi_\infty)$ , subject to further matching conditions needed to ensure the smoothness of  $\mathbb{A}$  on  $X$  when the two edges  $\{z = 0\}$  and  $\{z = 2\pi/\mu_0\}$  are glued together.

The simplest example of this correspondence is of course the case that the path in  $\mathcal{A}$  is constant so that  $c$  is identically 1 and  $c_2(\mathbb{E}, f)[X] = 0$ . Then we say that  $\mathbb{A}$  is the *pull-back of a monopole*. Here is a sort of converse:

**Proposition 1.** *Let  $\mathbb{A}$  be a framed caloron in a framed bundle with  $c_2(\mathbb{E}, f)[X] = 0$ . Then there is a deformation  $\mathbb{B}$  of  $\mathbb{A}$  (through framed caloron configurations), such that  $\mathbb{B}$  is the pull-back of a monopole.*

*Proof.* Since  $c_2(\mathbb{E}, f)[X] = 0$ , we can find a unitary automorphism  $C$  of  $\mathbb{E}$  over  $R$  which is equal to 1 on  $\{0\} \times \overline{B}^3$  and  $[0, 2\pi/\mu_0] \times S_\infty^2$ , and equal to  $c$  on  $\{2\pi/\mu_0\} \times \overline{B}^3$ . Pulling  $\mathbb{A}$  back by  $C$ , we reduce to the case  $c = 1$ , so that the caloron configuration is now a *loop* in  $\mathcal{A}$ . But this space is contractible, so the result follows.  $\square$

**2.2. Chern-Weil theory for framed bundles.** Another way to think about the invariant  $c_2(\mathbb{E}, f)[X]$  is in terms of the integral

$$\int_X \text{ch}(\mathbb{E}) = -\frac{1}{8\pi^2} \int_X \text{tr } F_{\mathbb{A}} \wedge F_{\mathbb{A}}, \quad (9)$$

where  $\mathbb{A}$  is some framed caloron configuration. If  $X$  had no boundary, this integral would give minus the second Chern class, but here there are additional contributions from  $\partial X$ . This integral has been calculated by different means in [8] and [9]. As in the previous section, we work over  $R$ , so that  $\mathbb{E}$  is identified with  $p^*E$ , together with a clutching function  $c$ . Using the familiar trick of writing

$$\text{tr } F_{\mathbb{A}} \wedge F_{\mathbb{A}} = d \text{tr } \left\{ d\mathbb{A} \wedge \mathbb{A} + \frac{2}{3} \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \right\}$$

the integral (9) becomes an integral over the boundary of the rectangle  $[0, 2\pi/\mu_0] \times \overline{B}^3$ :

$$-\frac{1}{8\pi^2} \int_X \text{tr } F_{\mathbb{A}} \wedge F_{\mathbb{A}} = -\frac{1}{8\pi^2} \int_{\partial([0, 2\pi/\mu_0] \times \overline{B}^3)} \text{tr } \left\{ d\mathbb{A} \wedge \mathbb{A} + \frac{2}{3} \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \right\}. \quad (10)$$

Now regard  $\mathbb{A}$  as a path  $(A_{(z)}, \Phi_{(z)})$  satisfying (7) and (8). Evaluating (10) on  $(\partial[0, 2\pi/\mu_0]) \times \overline{B}^3$  and using the clutching formulas gives

$$\begin{aligned} & -\frac{1}{8\pi^2} \int_{(\partial[0, 2\pi/\mu_0]) \times \overline{B}^3} \text{tr } \left\{ d\mathbb{A} \wedge \mathbb{A} + \frac{2}{3} \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \right\} = \\ & \quad -\frac{1}{24\pi^2} \int_{\overline{B}^3} \text{tr } (dcc^{-1})^3 + \frac{1}{8\pi^2} \int_{\overline{B}^3} d \text{tr } \{A(0)c^{-1}dc\}. \end{aligned}$$

The first term is  $\deg c = -c_2(\mathbb{E}, f)[X]$ , and the second vanishes because  $c = 1$  on  $S_{\infty}^2$ . On the other piece of the boundary we obtain

$$\begin{aligned} & -\frac{1}{8\pi^2} \int_{[0, 2\pi/\mu_0] \times S_{\infty}^2} \text{tr } \left\{ d\mathbb{A} \wedge \mathbb{A} + \frac{2}{3} \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \right\} = \\ & -\frac{1}{8\pi^2} \int_{[0, 2\pi/\mu_0] \times S_{\infty}^2} \text{tr } \{2F_A \wedge \Phi dz - dA \wedge \Phi dz + A \wedge d\Phi \wedge dz + \partial_z A \wedge A \wedge dz\}. \end{aligned}$$

The final term vanishes because the restriction of  $A$  to  $[0, 2\pi/\mu_0] \times S_{\infty}^2$  is pulled back from  $S_{\infty}^2$  so that  $\partial_z A = 0$  there (condition (i) of Definition 2). On the other hand, the sum of the middle two terms is exact, so does not contribute to the integral. The condition that  $A_{\infty}$  is compatible with  $\Phi_{\infty}$  implies that  $A_{\infty}$  decomposes as a direct sum of connections, one on each eigenbundle of  $E_{\infty}$ . Suppose  $E_{\mu}$  is the eigenbundle of  $\Phi_{\infty}$  with eigenvalue  $i\mu$ . Then  $A_{\infty} = \bigoplus a_{\mu}$  where  $a_{\mu}$  is a connection on  $E_{\mu}$ , and  $F_A|_{S_{\infty}^2} = \bigoplus f_{\mu}$  where  $f_{\mu}$  is the curvature of  $a_{\mu}$ . Since the first Chern class of  $E_{\mu}$  is given by

$$c_1(E_{\mu})[S_{\infty}^2] = \frac{i}{2\pi} \int_{S_{\infty}^2} \text{tr } f_{\mu}, \quad (11)$$

we have

$$-\frac{1}{8\pi^2} \int_{[0, 2\pi/\mu_0] \times S_{\infty}^2} \text{tr } 2F_A \wedge \Phi dz = -\frac{1}{\mu_0} \sum_{\mu} \mu c_1(E_{\mu})[S_{\infty}^2].$$

Putting the terms together, we arrive at the expression

$$\int_X \text{ch}(\mathbb{E}) = -c_2(\mathbb{E}, f)[X] - \frac{1}{\mu_0} \sum_{\mu} \mu c_1(E_{\mu})[S_{\infty}^2]. \quad (12)$$

## 3. BOUNDARY CONDITIONS VERSUS ASYMPTOTICS

In previous work, calorons have been studied exclusively as connections over  $S^1 \times \mathbb{R}^3$ , with decay conditions imposed near  $\infty$ ; the compact manifold  $X$  was not used. The purpose of this section is to show how the boundary conditions that are implicit in Definition 2 translate into the ‘BPS’ decay conditions for calorons that were written down in [9].

In order to compare the asymptotic region of  $S^1 \times \mathbb{R}^3$  with a neighbourhood of the boundary of  $X$ , choose polar coordinates  $r, y_1, y_2$  in  $\mathbb{R}^3$ , and continue to use  $z$  as a coordinate in  $S^1$ . Thus  $r$  is the distance from the origin in  $\mathbb{R}^3$  and  $y_1, y_2$  are some local angular coordinates on  $S_\infty^2$ . We suppose  $y_1$  and  $y_2$  are chosen so that  $g$  takes the form

$$g = dr^2 + r^2(h_1 dy_1^2 + h_2 dy_2^2) + dz^2,$$

for some positive locally-defined functions  $h_1, h_2$ . Local coordinates near the boundary of  $X$  will be  $x = r^{-1}$ ,  $y_1, y_2$  and  $z$ , so that  $x$  becomes a boundary defining function:  $x \geq 0$  on  $X$ , with equality only at  $\partial X$ . Writing  $g$  in terms of  $x$ ,

$$g = \frac{dx^2}{x^4} + h_1 \frac{dy_1^2}{x^2} + h_2 \frac{dy_2^2}{x^2} + dz^2.$$

Now denote the components of  $\mathbb{A}$ , in some gauge (trivialisation) that is smooth up to the boundary, by

$$\nabla_x = \partial_x + A_x, \nabla_{y_j} = \partial_{y_j} + A_{y_j}, \nabla_z = \partial_z + \Phi.$$

Performing a 3 + 1-decomposition of  $\mathbb{A} = (A_{(z)}, \Phi_{(z)})$  as before, we have, near the boundary,

$$\begin{aligned} \|A\|^2 &= |A_x|^2 |dx|_g^2 + |A_{y_1}|^2 |dy_1|_g^2 + |A_{y_2}|^2 |dy_2|_g^2 \\ &= x^2(x^2 |A_x|^2 + h_1^{-1} |A_{y_1}|^2 + h_2^{-1} |A_{y_2}|^2). \end{aligned}$$

Since  $x = r^{-1}$ , we see that  $\|A\| = O(r^{-1})$  as  $r \rightarrow \infty$ , uniformly in the angular variables  $(y_1, y_2)$ . This statement is not gauge invariant: a better formulation is that there exist preferred gauges near  $\infty$  in  $X^o$  (namely those that extend smoothly to  $\partial X$ ), such that the connection 1-form satisfies  $\|A\| = O(r^{-1})$  in such a gauge. In such gauges we also have  $\|\Phi\| = O(1)$ .

Similarly, the assumptions in Definition 2 imply that  $\nabla_{y_j} \Phi$  and  $\partial_z A_{y_j}$  are  $O(x)$  as  $x \rightarrow 0$ , while  $\nabla_x \Phi$  and  $\partial_z A_x$  are  $O(1)$ . It follows that  $\|\nabla_A \Phi - \partial_z A\| = O(r^{-2})$  as  $r \rightarrow \infty$ , a fact that will be used in the next section.

## 4. PROOF OF THEOREM 1.1

In this section we give two proofs of Theorem 1 about the Fredholm properties of  $D_\mathbb{A}^+$ . The first proof rests on a result of Anghel in [1], the second on the general theory of  $\Phi$ -pseudodifferential operators. Of course both methods give the same answer, and indeed the key point is the same in each case.

**4.1. First proof.** Theorem (2.1) of [1] gives conditions for  $D_\mathbb{A} = D_\mathbb{A}^+ \oplus D_\mathbb{A}^-$  to be Fredholm:  $D_\mathbb{A}$  is Fredholm if and only if there is a compact set  $K \subset X^o$  and a constant  $C > 0$  such that

$$\|D_\mathbb{A} \psi\|_{L^2} \geq C \|\psi\|_{L^2}, \text{ when } \psi \in W^1(S \otimes \mathbb{E}) \text{ and } \text{Supp}(\psi) \subset X^o \setminus K.$$

If  $D_\mathbb{A}$  is Fredholm then  $D_\mathbb{A}^+$  must be Fredholm too. Now

$$D_\mathbb{A}^* D_\mathbb{A} = (D_\mathbb{A}^- D_\mathbb{A}^+) \oplus (D_\mathbb{A}^+ D_\mathbb{A}^-)^*$$

so to obtain estimates on  $\|D_{\mathbb{A}}\|_{L^2}$  we consider the operator  $D_{\mathbb{A}}^- D_{\mathbb{A}}^+$ . Using the notation and conventions of §1.1 we have from (2),

$$D_{\mathbb{A}}^- D_{\mathbb{A}}^+ = -\nabla_z^2 + [D_A, \nabla_z] + D_A^2.$$

The third term here is clearly positive, and the boundary conditions allow us to estimate the other two as follows.

**The first term.** Extend the framing  $f$  to a neighbourhood of  $\partial X$ ; this gives a gauge near  $\infty$  in which the ‘3+1’ decomposition (6) can be performed. As the boundary  $\partial X$  is approached the eigenvalues of  $\Phi$  converge to the eigenvalues of  $\Phi_\infty$ . Using spherical polar coordinates on  $\mathbb{R}^3$ , let  $i\lambda_j(r, y_1, y_2, z)$  be the eigenvalues of  $\Phi$ , and  $i\mu_j$  be the eigenvalues of  $\Phi_\infty$  ( $j = 1, \dots, n$ ) such that  $\lambda_j \rightarrow \mu_j$  as  $r \rightarrow \infty$ . Let  $\lambda(r, y_1, y_2, z)$  be the smallest element in  $\{|\lambda_j + k\mu_0| : j = 1, \dots, n; k \in \mathbb{Z}\}$  and  $\mu$  be the smallest element of the set  $\{|\mu_j + k\mu_0| : j = 1, \dots, n, k \in \mathbb{Z}\}$ . The invertibility condition on  $\Phi_\infty$  in the statement of the theorem implies that  $\mu > 0$ , so there exists a compact set  $K_1 \subset X^o$  such that  $\lambda > \mu/2$  on  $X^o \setminus K_1$ .

Suppose  $\psi \in W^2(S^+ \otimes \mathbb{E})$  and  $\text{Supp } \psi \subset X^o \setminus K_1$ . Using the isomorphism  $S^+ \cong p^*S_{(3)}$ ,  $\psi$  can be written as a Fourier series

$$\psi = \sum_k \exp(ik\mu_0 z) \phi_k$$

where  $\phi_k$  is a section of  $S_{(3)} \otimes E$ . Let

$$\psi^{(k)} = \exp(ik\mu_0 z) \phi_k.$$

Then

$$\nabla_z \psi^{(k)} = (ik\mu_0 + \Phi) \psi^{(k)}$$

so

$$(-\nabla_z \nabla_z \psi^{(k)}, \psi^{(k)}) \geq \frac{1}{4} \mu^2 \|\psi^{(k)}\|^2, \quad \text{on } X^o \setminus K_1$$

as a pointwise estimate. (Since  $\psi^{(k)} \in W^2(S^+ \otimes \mathbb{E})$ ,  $\psi^{(k)}$  is actually continuous so both sides of the inequality exist.) Since the inequality is independent of  $k$  it holds for general  $\psi$  and we obtain

$$\text{Supp}(\psi) \subset X^o \setminus K_1 \Rightarrow (-\nabla_z \nabla_z \psi, \psi)_{L^2} \geq \frac{1}{4} \mu^2 \|\psi\|_{L^2}^2. \quad (13)$$

**The second term.** We have

$$[D_A, \nabla_z] = \sum_j e_j [\nabla_j, \nabla_z] = \sum_j \iota(\partial_j) (\nabla_A \Phi - \partial_z A)$$

where  $\iota(\xi)$  denotes interior product with  $\xi$ . But  $\|\nabla_A \Phi - \partial_z A\| \rightarrow 0$  as  $r \rightarrow \infty$ , so there exists a compact set  $K_2 \subset X^o$  such that

$$\text{Supp}(\psi) \subset X^o \setminus K_2 \Rightarrow |([D_A, \nabla_z] \psi, \psi)_{L^2}| \leq \frac{1}{8} \mu^2 \|\psi\|_{L^2}^2. \quad (14)$$

Now let  $K$  be a compact set containing  $K_1$  and  $K_2$ . Combining (13) and (14) we obtain

$$\text{Supp}(\psi) \subset X^o \setminus K \Rightarrow (D_{\mathbb{A}}^- D_{\mathbb{A}}^+ \psi, \psi)_{L^2} \geq \frac{1}{8} \mu^2 \|\psi\|_{L^2}^2.$$

A similar bound is obtained for  $D_{\mathbb{A}}^+ D_{\mathbb{A}}^- = (D_{\mathbb{A}}^- D_{\mathbb{A}}^+)^*$ , and so we obtain the following bound for  $D_{\mathbb{A}}$ :

$$\psi \in W^2(S \otimes \mathbb{E}), \text{Supp}(\psi) \subset X^o \setminus K \Rightarrow \|D_{\mathbb{A}} \psi\|_{L^2} \geq \frac{1}{\sqrt{8}} \mu \|\psi\|_{L^2}.$$

By density, the inequality in fact holds for  $\psi \in W^1(S \otimes \mathbb{E})$ . This completes the verification of Anghel’s criterion and gives a proof of the ‘if’ part of Theorem 1.



The ‘only if’ part can also be proved in this framework but this is omitted. This converse statement also follows at once from the discussion of the next section.

**4.2. Second proof.** Recall the boundary-adapted coordinate system  $x, y_1, y_2, z$  introduced in §3, and let the components of  $\nabla_{\mathbb{A}}$  in these coordinates be

$$\nabla_x = \partial_x + A_x, \nabla_{y_j} = \partial_{y_j} + A_{y_j}, \nabla_z = \partial_z + A_z.$$

Relative to a suitable choice of basis for the spin-bundles, we have then

$$D_{\mathbb{A}}^+ = \nabla_z + e_1 x \nabla_{y_1} + e_2 x \nabla_{y_2} + e_3 x^2 \nabla_x.$$

Strictly speaking, we are making a choice of normal coordinates here; otherwise there will be additional zero-order terms coming from connection coefficients. This is an example of a  $\Phi$ -differential operator in the sense of [14]; more generally the algebra of  $\Phi$ -differential operators on  $X$  consists of all differential operators which take the form

$$P(x, y, z; x^2 \partial_x, x \partial_y, \partial_z), \quad (15)$$

near  $\partial X$ , where  $P$  is smooth in the first three variables and polynomial in the last three variables. In [14] it is shown that such an operator is Fredholm in  $L^2$  if and only if it is *fully elliptic* in the following sense. First, (15) must be elliptic in the usual sense over  $X^o$ . Secondly, the associated *indicial family* must be invertible on every fibre  $p^{-1}(y) \subset \partial X$ . Given such a fibre, the indicial family on  $p^{-1}(y)$  is defined by picking a real number  $\xi$  and a real cotangent vector  $\eta \in T_y^* S_\infty^2$ , and defining

$$\hat{P}_{(y, \eta, \xi)} = P(0, y, z; i\xi, i\eta, \partial_z)$$

as a differential operator on  $p^{-1}(y)$ . To say that the indicial family is invertible is to say that  $\hat{P}_{(y, \eta, \xi)}$  is invertible (in any reasonable space of sections over  $p^{-1}(y)$ ), for each choice of  $(y, \eta, \xi)$  as above.

Following this recipe for  $D_{\mathbb{A}}^+$ , we obtain

$$\hat{P}_{(y, \eta, \xi)} = \nabla_z + i(\eta_1 e_1 + \eta_2 e_2 + \xi e_3).$$

This operator in  $C^\infty(S^1, p^* S_{(3)} \otimes E_\infty)$  is a sum of two terms  $B + A$ , where  $A = i(\eta_1 e_1 + \eta_2 e_2 + \xi e_3)$  is self-adjoint,  $B = \nabla_z$  is skew-adjoint and  $[A, B] = 0$ . It follows by considering  $(A+B)^*(A+B)$  that  $(A+B)u = 0$  if and only if  $Au = 0$  and  $Bu = 0$ . Now  $B$  has a non-trivial null-space only if one of the  $\mu_j$  is an integral multiple of  $\mu_0$ . Hence under the assumption of Theorem 1,  $A + B$  is injective. Similarly the adjoint  $(A+B)^* = A - B$  is injective, so that the hypothesis of Theorem 1 implies that the indicial family is invertible, and so  $D_{\mathbb{A}}^+$  is Fredholm in  $L^2$ . Conversely, if the condition fails, then  $B$  is not invertible, and nor is  $B + A$  when  $\eta_j = 0 = \xi$ . So in this case  $D_{\mathbb{A}}^+$  is not fully elliptic and hence cannot be Fredholm in  $L^2$ . The proof of Theorem 1 is now complete.

**4.3. Remarks about the  $L^2$ -condition.** According to [14, Proposition 9], elements of the null-space of a fully elliptic  $\Phi$ -differential operator decay very rapidly at the boundary. More precisely, if  $D_{\mathbb{A}}^+$  is fully elliptic, if  $D_{\mathbb{A}}^+ \psi = 0$ , and if for some real  $m$ ,  $x^m \psi \in L^2(X)$ , then  $\psi \in C^\infty(X)$  and  $\psi$  vanishes to all orders in  $x$  at  $\partial X$ . There is a similar statement for the cokernel. Now in terms of the boundary-adapted coordinates  $(x, y_j, z)$ , the volume element determined by the metric  $g$  has the form  $x^{-4} d\mu$  where  $d\mu = h_1 h_2 dx dy_1 dy_2 dz$ . It follows from the above that the index of (3) is the same as the index of

$$D_{\mathbb{A}}^+ : W^1(X, \mathbb{E} \otimes S^+, d\mu) \rightarrow W^0(X, \mathbb{E} \otimes S^-, d\mu). \quad (16)$$

This fact makes the next result almost obvious:

**Proposition 2.** *Let  $\mathbb{A}, \mathbb{B}$  be two caloron configurations on  $(\mathbb{E}, f)$ , both framed by  $(A_\infty, \Phi_\infty)$ . Then  $D_{\mathbb{A}}^+$  is Fredholm if and only if  $D_{\mathbb{B}}^+$  is so, and their  $L^2$ -indices coincide.*

*Proof.* The space of calorons on a given framed bundle, with given boundary data  $(A_\infty, \Phi_\infty)$  is contractible. It is easy to see that any continuous path joining  $\mathbb{A}$  to  $\mathbb{B}$  gives rise to a norm-continuous path of Dirac operators between the Sobolev spaces in (16). Since each of these is Fredholm by Theorem 1, it follows that the index is constant on this path.  $\square$

## 5. PROOF OF THE INDEX THEOREM

**5.1. Proof when  $c_2(\mathbb{E}, f)[X] = 0$ .** In this case, by Propositions 1 and 2 it is enough to compute the index when  $\mathbb{E} = p^*(E)$  and  $\mathbb{A} = p^*A + p^*\Phi dz$  is the pull-back of a monopole (cf. §2.1). Then the coefficients of  $D_{\mathbb{A}}^+$  are independent of  $z$  and we can use Fourier analysis in the  $S^1$ -variable to reduce the calculation of the index to that of a collection of operators of the form (1) on  $\mathbb{R}^3$ . These operators are precisely the subject of the CAR theorem in its simplest form [7].

Let

$$Y_k = \{\psi = \exp(ik\mu_0 z)\phi : \phi \in W^0(\mathbb{R}^3, S_{(3)} \otimes E)\}$$

so that

$$W^0(S^+ \otimes \mathbb{E}) = \left\{ \sum \psi^{(k)} : \psi^{(k)} \in Y_k \text{ and } \sum \|\psi^{(k)}\|^2 < \infty \right\}.$$

Since by assumption the coefficients are independent of  $z$ ,  $D_{\mathbb{A}}^+$  maps  $Y_k \cap W^1$  into  $Y_k$  and its restriction to this subspace is equal to

$$\begin{aligned} D_k : W^1(S_{(3)} \otimes E) &\longrightarrow W^0(S_{(3)} \otimes E) \\ D_k &= D_A + ik\mu_0 + 1 \otimes \Phi. \end{aligned}$$

According to the theory developed by Callias–Anghel–Råde,  $D_k$  is Fredholm for every  $k \in \mathbb{Z}$  iff  $D_{\mathbb{A}}^+$  is Fredholm, and [17] shows that:

$$\begin{aligned} \text{ind } D_k &= - \int_{S_\infty^2} \hat{A}(S_\infty^2) \wedge \text{ch}(E_{(k)}^+) \\ &= -c_1(E_{(k)}^+)[S_\infty^2] \end{aligned}$$

where  $E_{(k)}^+$  is the subbundle of  $E_\infty$  on which  $(k\mu_0 - i\Phi_\infty)$  has positive eigenvalues. (We have already noted that this sum is finite.) Since  $Y_j \cap Y_k = 0$  if  $j \neq k$ , the index of  $D_{\mathbb{A}}^+$  is the sum of the indices of the  $D_k$ , i.e.

$$\text{ind } D_{\mathbb{A}}^+ = \sum_k \text{ind } D_k = - \sum_k c_1(E_{(k)}^+)[S_\infty^2].$$

That completes the proof of Theorem 2 when  $c_2(\mathbb{E}, f)[X] = 0$ .

**5.2. Proof of the index theorem when  $c_2(\mathbb{E}, f)[X] \neq 0$ .** Anghel [1], generalizing work of Gromov and Lawson [11], has given an excision theorem which compares the  $L^2$ -indices of a pair of Dirac operators over a complete manifold that agree near infinity. In our case this result yields the following statement. Let  $\mathbb{E}$  and  $\mathbb{F}$  be a pair of bundles over  $X^\circ$  and let  $\mathbb{A}$  and  $\mathbb{B}$  be unitary connections on  $\mathbb{E}$  and  $\mathbb{F}$  (respectively). Suppose that there is a bundle isometry  $\theta : \mathbb{E}|_{X^\circ \setminus K} \rightarrow \mathbb{F}|_{X^\circ \setminus K}$  which carries  $\mathbb{A}$  to  $\mathbb{B}$  outside some compact set  $K \subset X^\circ$ . Then

$$\text{ind } D_{\mathbb{A}}^+ - \text{ind } D_{\mathbb{B}}^+ = \int_{X^\circ} \text{ch}(\mathbb{E}) - \int_{X^\circ} \text{ch}(\mathbb{F}). \quad (17)$$

We are going to deduce Theorem 2 by taking for  $\mathbb{B}$  a connection which agrees with  $\mathbb{A}$  near  $\infty$ , but which lives on a framed bundle  $(\mathbb{F}, f)$  with  $c_2(\mathbb{F}, f) = 0$ . This will complete the proof in view of the results of the previous section.

Let then  $(\mathbb{E}, f)$  be a framed bundle and  $\mathbb{A}$  a framed caloron configuration on  $\mathbb{E}$ . As in §2.1, identify  $\mathbb{E}$  with  $p^*E$ , together with a clutching function  $c \in \mathcal{C}$ . Extend the framing smoothly from the boundary to a region  $[0, 2\pi/\mu_0] \times U$  where  $K \subset \mathbb{R}^3$  is compact and  $U = X^\circ \setminus K$ . By a deformation of  $\mathbb{A}$  over  $\{2\pi/\mu_0\} \times U$  which vanishes at  $\infty$ , we can assume that  $c = 1$  on  $U$ . Now define  $\mathbb{F} = p^*E$  and  $\mathbb{B}$  to agree with  $\mathbb{A}$  over  $S^1 \times U$ , but extended over  $S^1 \times K$  to define a smooth connection on  $\mathbb{F}$ . (This can be achieved by a suitable use of cut-off functions.)

Applying (17),

$$\text{ind } D_{\mathbb{A}}^+ - \text{ind } D_{\mathbb{B}}^+ = \int_{X^\circ} \text{ch}(\mathbb{E}) - \int_{X^\circ} \text{ch}(\mathbb{F}).$$

But

$$\int_{X^\circ} \text{ch}(\mathbb{E}) = -c_2(\mathbb{E}, f)[X] - \frac{1}{\mu_0} \sum_{\mu} \mu c_1(E_{\mu})[S_{\infty}^2]$$

from (12), and

$$\int_{X^\circ} \text{ch}(\mathbb{F}) = -\frac{1}{\mu_0} \sum_{\mu} \mu c_1(E_{\mu})[S_{\infty}^2].$$

So

$$\text{ind } D_{\mathbb{A}}^+ = \text{ind } D_{\mathbb{B}}^+ - c_2(\mathbb{E}, f)[X]$$

From §5.1 we know that  $\text{ind } D_{\mathbb{B}}^+ = -\sum_k c_1(E_{(k)}^+)[S_{\infty}^2]$  so we have proved that

$$\text{ind } D_{\mathbb{A}}^+ = -c_2(\mathbb{E}, f)[X] - \sum_k c_1(E_{(k)}^+)[S_{\infty}^2].$$

This completes the proof of Theorem 2.

#### APPENDIX A. ADIABATIC LIMITS OF $\eta$ -INVARIANTS

The index formula (4) can be re-expressed in a form reminiscent of the APS formula by using adiabatic limits of  $\eta$ -invariants. The aim of this Appendix is to explain roughly how this can be done, thereby sketching a proof of Corollary 1.

First recall the APS formula for the index of a Dirac operator  $D$  on a manifold  $X$  with cylindrical end [4, Theorem 3.10]:

$$\text{ind } D = \int_X \alpha_0 - \frac{h + \eta(D_{\partial})}{2}. \quad (18)$$

The first term is the integral over  $X$  of some form  $\alpha_0$ , while the second depends on a Dirac operator  $D_{\partial}$  associated to the boundary  $\partial X$ :  $h = \dim \ker D_{\partial}$  and  $\eta(D_{\partial})$  is the  $\eta$ -invariant of  $D_{\partial}$ . The  $\eta$ -invariant measures the asymmetry of the spectrum of  $D_{\partial}$ , and is defined to be  $\eta(D_{\partial}) = \eta(D_{\partial})(0)$  where:

$$\eta(D_{\partial})(s) = \sum_{\substack{\lambda \in \text{Spec } D_{\partial} \\ \lambda \neq 0}} (\text{sign } \lambda) |\lambda|^{-s}.$$

Part of the proof of the APS theorem involves showing  $\eta(D_{\partial})(s)$  is analytic at  $s = 0$ , so that the definition  $\eta(D_{\partial}) = \eta(D_{\partial})(0)$  makes sense.

Next consider a closed manifold  $M$  equipped with a family of metrics  $g_{\epsilon}$ , where part of the metric blows up as  $\epsilon \rightarrow 0$ , and suppose that  $D_{\epsilon}$  is a family of Dirac operators on  $(M, g_{\epsilon})$ , with  $\eta$ -invariant  $\eta(D_{\epsilon})$ . Bismut and Cheeger [5] show that, under certain conditions, not only does  $\eta(D_{\epsilon})$  exist for each  $\epsilon > 0$ , but remarkably,

the limit  $\lim_{\epsilon \rightarrow 0} \eta(D_\epsilon)$  exists. They consider fibrations  $Z \rightarrow M \xrightarrow{p} B$  of compact oriented spin manifolds and equip  $M$  with a family of metrics

$$g_\epsilon = \epsilon^{-1} p^*(g^B) + g^Z$$

where  $g^B$  is a metric on  $B$  and  $g^Z$  annihilates the orthogonal complement of the fibres. When  $D_\epsilon$  is a Dirac operator on  $(M, g_\epsilon)$  coupled to some auxilliary bundle, Bismut and Cheeger [5] show the limit of the reduced  $\eta$ -invariant,  $\lim_{\epsilon \rightarrow 0} \bar{\eta}(D_\epsilon)$  exists. (The reduced  $\eta$ -invariant of an operator  $D$  is  $\bar{\eta}(D) = (h + \eta(D))/2$ , which corresponds to the second term of the APS formula (18).) Moreover, they show that the limit is given by the integral of some form  $\hat{\eta}$  over the base  $B$ , and give explicit formulae for  $\hat{\eta}$ .

We apply this theory to the fibration  $S^1 \rightarrow S^1 \times S_\infty^2 \xrightarrow{p} S_\infty^2$ , where the metric on  $M = S^1 \times S_\infty^2$  is given by

$$g_\epsilon = \epsilon^{-1}(h_1 dy_1^2 + h_2 dy_2^2) + dz^2$$

in the notation of Section 3. Of course, this is the same as the restriction of the metric  $g$  on  $S^1 \times \bar{B}^3$  to  $\chi^2 = \epsilon$ . Let  $D_\epsilon$  be the Dirac operator on  $(M, g_\epsilon)$  coupled to the bundle  $p^*E_\infty$  via the connection  $p^*A_\infty + p^*\Phi_\infty dz$ . our aim is to calculate the limit  $\bar{\eta}_{\text{lim}} = \lim_{\epsilon \rightarrow 0} \bar{\eta}(D_\epsilon)$ . According to [5, Theorem 4.95] this is given by an integral

$$\bar{\eta}_{\text{lim}} = \frac{1}{2\pi i} \int_{S_\infty^2} \hat{\eta}.$$

The definition of  $\hat{\eta}$  [5, Definition 4.93] simplifies in our situation to

$$\hat{\eta} = \frac{1}{\pi^{1/2}} \int_0^\infty \text{tr}^{\text{even}} (D_Z \exp(-F_\infty - u D_Z^2)) \frac{du}{2u^{1/2}} \quad (19)$$

where  $F_\infty$  is the curvature of  $A_\infty$  and  $D_Z$  is the Dirac operator associated to the fibre  $Z = S^1$ , which is given by

$$D_Z = -i \frac{d}{dz} - i\Phi_\infty.$$

Recall the notation of Section 2.2:  $\Phi_\infty$  decomposes  $E_\infty$  into a direct sum of eigen-bundles  $E_\mu = \bigoplus E_\mu$ . The Fredholm condition (Theorem 1) implies that each eigenvalue  $i\mu$  of  $\Phi_\infty$  can be written as  $\mu = N_\mu \mu_0 + \epsilon_\mu$  where  $N_\mu \in \mathbb{Z}$  and  $0 < \epsilon_\mu < \mu_0$ .  $D_Z$  decomposes as  $D_Z = \bigoplus D_Z^{(\mu)}$ , where

$$D_Z^{(\mu)} = -i \frac{d}{dz} + \mu$$

and  $A_\infty$  decomposes as a direct sum of connections on each eigen-bundle,  $A_\infty = \bigoplus a_\mu$ . Then

$$\begin{aligned} \bar{\eta}_{\text{lim}} &= \frac{1}{2\pi i} \int_{S_\infty^2} \frac{1}{\pi^{1/2}} \int_0^\infty \text{tr}^{\text{even}} (D_Z \exp(-F_\infty - u D_Z^2)) \frac{du}{2u^{1/2}} \\ &= \sum_\mu \left( \frac{i}{2\pi} \int_{S_\infty^2} f_\mu \right) \left( \frac{1}{\pi^{1/2}} \int_0^\infty \text{tr} [D_Z^{(\mu)} \exp(-u (D_Z^{(\mu)})^2)] \frac{du}{2u^{1/2}} \right) \end{aligned}$$

where  $f_\mu$  is the curvature of  $a_\mu$ . Using equation (11) the first bracket is  $c_1(E_\mu)[S_\infty^2]$  while the second is the  $\eta$ -invariant of  $D_Z^{(\mu)}$  which we denote  $\eta_\mu$  (this follows from [5, Equation 0.3] or [6, Theorem 2.6]). Hence

$$\bar{\eta}_{\text{lim}} = \sum_\mu \eta_\mu c_1(E_\mu)[S_\infty^2].$$

Calculating  $\eta_\mu$  is a standard example (see [3] or [10, Section 1.10]):

$$\eta_\mu = 1 - \frac{2\epsilon_\mu}{\mu_0}$$

and so

$$\bar{\eta}_{\text{lim}} = -\frac{2}{\mu_0} \sum_{\mu} \epsilon_\mu c_1(E_\mu)[S_\infty^2]. \quad (20)$$

since

$$\sum_{\mu} c_1(E_\mu)[S_\infty^2] = 0. \quad (21)$$

Using (12), the index formula (4) can be written as

$$\text{ind}(D_{\mathbb{A}}^+) = \int_X \text{ch}(\mathbb{E}) + \frac{1}{\mu_0} \sum_{\mu} \mu c_1(E_\mu)[S_\infty^2] - \sum_k c_1(E_{(k)}^+)[S_\infty^2].$$

The last two terms together become

$$\begin{aligned} \frac{1}{\mu_0} \sum_{\mu} \mu c_1(E_\mu)[S_\infty^2] - \sum_k c_1(E_{(k)}^+)[S_\infty^2] &= \frac{1}{\mu_0} \sum_{\mu} \epsilon_\mu c_1(E_\mu)[S_\infty^2] \\ &\quad + \left( \sum_{\mu} N_{\mu} c_1(E_\mu)[S_\infty^2] - \sum_k \sum_{\mu: N_{\mu} \geq -k} c_1(E_\mu)[S_\infty^2] \right). \end{aligned}$$

The term in brackets vanishes because of identity (21), so

$$\begin{aligned} \text{ind}(D_{\mathbb{A}}^+) &= \int_X \text{ch}(\mathbb{E}) + \frac{1}{\mu_0} \sum_{\mu} \epsilon_\mu c_1(E_\mu)[S_\infty^2] \\ &= \int_X \text{ch}(\mathbb{E}) - \frac{1}{2} \bar{\eta}_{\text{lim}} \end{aligned} \quad (22)$$

using equation (20). Thus we have proved Corollary 1. In this form the index formula resembles the APS formula (18) and it seems likely that equation (22) might apply more widely to Dirac operators on manifolds with fibred boundaries.

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